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Ph.D. DISSERTATION DEFENSE

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Title: Quadratic equations in wreath products of abelian groups

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ABSTRACT

This work focuses on the computational complexity of the Diophantine problem for orientable and non-orientable quadratic equations over wreath products of finitely generated abelian groups. The Diophantine problem for quadratic equations naturally generalizes fundamental (Dehn) problems of group theory such as the word and conjugacy problems. Furthermore, there is a deep relationship between quadratic equations and compact surfaces, which makes quadratic equations an interesting object of study. In this work, we aim to provide a complete classification of cases, depending on the relationship between characteristics of the top group and characteristics (such as genus) of a given equation, when the Diophantine problem is **NP**-complete or polynomial-time decidable.

We prove that, over the lamplighter group, the Diophantine problem for spherical equations is **NP**-complete and both the conjugacy problem and the Diophantine problem for orientable quadratic equations are decidable in linear time. We show that the Diophantine problem for genus two non-orientable quadratic equations over the lamplighter group is decidable in linear time and for genus one is **NP**-complete. Next, we show that for a general wreath product of finitely generated abelian groups A and B, the Diophantine problem for orientable equations remains in **NP** even when A and B are given as part of the input. However, in the general case, parameters such as the ranks of A and B and the genus of the equation impact the complexity of solving these equations. The complexity of the problem is particularly influenced by the group B. Specifically, for a fixed $A \wr B$, the problem is **NP**-hard if and only if $|B| = \infty$. The group A's impact on the complexity is limited. However, if A is a part of the input, the problem becomes **NP**-hard even when $B = \mathbb{Z}_2$ is fixed. The genus g also plays a crucial role in the complexity. If g is sufficiently large, then the equation can be solved in polynomial time. Otherwise, for intermediate values of g, the problem generally remains **NP**-hard.